Changing the Basis of an Image

Worksheet solutions

Answers are shown in red.

In this worksheet you will rewrite a small image using a different basis. When every coefficient is kept, the change of basis is **lossless**: you get the original image back exactly. When the smallest coefficients are dropped, it is **lossy**: the image is only approximated, but often surprisingly well.

Part 1: A 2×2 Image

Take this 2×2 grayscale image:

Image

10	6
4	4

The standard basis

The “normal” way of storing an image is already a basis. Here are four basis images, each with a 1 in a different spot:

e1

1	0
0	0

e2

0	1
0	0

e3

0	0
1	0

e4

0	0
0	1

Multiply each basis image by the pixel value in its spot and add them up:

10 · e1

10	0
0	0

+

6 · e2

0	6
0	0

+

4 · e3

0	0
4	0

+

4 · e4

0	0
0	4

=

Original

10	6
4	4

The result is exactly the original image. In the standard basis the coefficients are just the pixel values: 10, 6, 4, 4. But pixels are not the only option. Any set of four pattern images that can be combined to make every possible 2×2 image is a basis, and the same image can be described with different coefficients.

A new basis (2×2 Haar)

Here are four “pattern images”:

b1 (average)

1	1
1	1

b2 (top vs bottom)

1	1
-1	-1

b3 (left vs right)

1	-1
1	-1

b4 (diagonal)

1	-1
-1	1

These patterns are orthogonal and each has squared length 4, so each coefficient is just a dot product divided by 4: multiply the image and the pattern pixel by pixel, add the four products, and divide by 4.

Compute the coefficients

Show your dot products.

c_1 = $(10 + 6 + 4 + 4) / 4 = 24 / 4 = \mathbf{6}$

c_2 = $(10 + 6 - 4 - 4) / 4 = 8 / 4 = \mathbf{2}$

c_3 = $(10 - 6 + 4 - 4) / 4 = 4 / 4 = \mathbf{1}$

c_4 = $(10 - 6 - 4 + 4) / 4 = 4 / 4 = \mathbf{1}$

Lossless reconstruction

Multiply each basis image by its coefficient, then add the four results pixel by pixel. Compare your answer with the original image.

$c_1 \cdot$ b1			$c_2 \cdot$ b2			$c_3 \cdot$ b3			$c_4 \cdot$ b4			Result	
6	6	+	2	2	+	1	-1	+	1	-1	=	10	6
6	6		-2	-2		1	-1		-1	1		4	4

Does your reconstruction match the original exactly? **Yes.** Every pixel matches (top-left: 6 + 2 + 1 + 1 = 10, top-right: 6 + 2 − 1 − 1 = 6, and so on).

Lossy reconstruction

Now keep only the two largest coefficients, c_1 and c_2, and discard the other two. Rebuild the image from just those two terms.

$c_1 \cdot b1$			$c_2 \cdot b2$			Approximation	
6	6	+	2	2	=	8	8
6	6		-2	-2		4	4

Error (original minus approximation), pixel by pixel:

2	-2
0	0

Squared error (sum of squared errors) = $2^2 + (-2)^2 + 0^2 + 0^2 = \mathbf{8}$

$4 \times (c_3^2 + c_4^2) =$ $4 \times (1^2 + 1^2) = \mathbf{8}$

What do you notice? The two numbers are equal. The squared error from dropping coefficients is exactly 4 times the sum of the squares of the dropped coefficients, so dropping small coefficients produces only a small error.

Part 2: A 3×3 Image with the Frei-Chen Basis

A 3×3 image has nine pixels, so a basis needs nine pattern images. The Frei-Chen basis was designed with meaning in mind: four patterns respond to edges, four respond to lines, and one is the average.

The basis

Each mask is shown with integer-ish entries, followed by the number you divide by to give it length 1.

G1 (÷ 2√2)

1	√2	1
0	0	0
-1	−√2	-1

G2 (÷ 2√2)

1	0	-1
√2	0	−√2
1	0	-1

G3 (÷ 2√2)

0	-1	√2
1	0	-1
−√2	1	0

G4 (÷ 2√2)

√2	-1	0
-1	0	1
0	1	−√2

Edge patterns (G1–G4)

G5 (÷ 2)

0	1	0
-1	0	-1
0	1	0

G6 (÷ 2)

-1	0	1
0	0	0
1	0	-1

G7 (÷ 6)

1	-2	1
-2	4	-2
1	-2	1

G8 (÷ 6)

-2	1	-2
1	4	1
-2	1	-2

G9 (÷ 3)

1	1	1
1	1	1
1	1	1

Line patterns (G5–G8) and average (G9)

This basis is *orthonormal*: each coefficient is just the dot product with the normalized mask, with no extra division at the end.

The image

A bright horizontal line on a dark background:

Image

0	0	0
9	9	9
0	0	0

Compute the coefficients

Tip: take the dot product with the integer mask, then divide by the mask's divisor. Look at which rows of the image are nonzero before you start; it can save you a lot of work.

G1 coefficient = middle row of G1 is all 0 → **0**

G2 coefficient = $9\sqrt{2} + 0 - 9\sqrt{2} = 0$ → **0**

G3 coefficient = $9 + 0 - 9 = 0$ → **0**

G4 coefficient = $-9 + 0 + 9 = 0$ → **0**

G5 coefficient = $-9 + 0 - 9 = -18$, ÷ 2 → **−9**

G6 coefficient = middle row of G6 is all 0 → **0**

G7 coefficient = $-18 + 36 - 18 = 0$ → **0**

G8 coefficient = $9 + 36 + 9 = 54$, ÷ 6 → **9**

G9 coefficient = $9 + 9 + 9 = 27$, ÷ 3 → **9**

Only the middle row of the image is nonzero, so only the middle row of each mask matters.

Which coefficients are nonzero? What does that tell you about the image?

Only G5, G8 and G9 are nonzero. All four edge coefficients (G1–G4) are 0,

so the image has no edges; it is a line (plus an average brightness).

Lossless reconstruction

Keep only the nonzero coefficients. Multiply each one by its *normalized* basis image (the mask divided by its divisor), then add the results. Label each grid with the coefficient and basis image you used. You may not need every grid.

−9 × G5/2

0	−4.5	0
4.5	0	4.5
0	−4.5	0

+

9 × G8/6

-3	1.5	-3
1.5	6	1.5
-3	1.5	-3

+

9 × G9/3

3	3	3
3	3	3
3	3	3

+

(not needed)

=

Reconstructed

0	0	0
9	9	9
0	0	0

Does your reconstruction match the original exactly? **Yes.**

Work space

−9 × G5/2 = −4.5 × G5, 9 × G8/6 = 1.5 × G8, 9 × G9/3 = 3 × G9
Corners: 0 − 3 + 3 = 0 · top and bottom middle: −4.5 + 1.5 + 3 = 0
Middle left and right: 4.5 + 1.5 + 3 = 9 · center: 0 + 6 + 3 = 9