Changing the Basis of an Image

Student worksheet

Name: _____ Date: _____

In this worksheet you will rewrite a small image using a different basis. When every coefficient is kept, the change of basis is **lossless**: you get the original image back exactly. When the smallest coefficients are dropped, it is **lossy**: the image is only approximated, but often surprisingly well.

Part 1: A 2×2 Image

Take this 2×2 grayscale image:

Image

10	6
4	4

The standard basis

The “normal” way of storing an image is already a basis. Here are four basis images, each with a 1 in a different spot:

e1

1	0
0	0

e2

0	1
0	0

e3

0	0
1	0

e4

0	0
0	1

Multiply each basis image by the pixel value in its spot and add them up:

10 · e1

10	0
0	0

+

6 · e2

0	6
0	0

+

4 · e3

0	0
4	0

+

4 · e4

0	0
0	4

=

Original

10	6
4	4

The result is exactly the original image. In the standard basis the coefficients are just the pixel values: 10, 6, 4, 4. But pixels are not the only option. Any set of four pattern images that can be combined to make every possible 2×2 image is a basis, and the same image can be described with different coefficients.

A new basis (2×2 Haar)

Here are four “pattern images”:

b1 (average)

1	1
1	1

b2 (top vs bottom)

1	1
-1	-1

b3 (left vs right)

1	-1
1	-1

b4 (diagonal)

1	-1
-1	1

These patterns are orthogonal and each has squared length 4, so each coefficient is just a dot product divided by 4: multiply the image and the pattern pixel by pixel, add the four products, and divide by 4.

Compute the coefficients

Show your dot products.

c_1 = _____

c_2 = _____

c_3 = _____

c_4 = _____

Lossless reconstruction

Multiply each basis image by its coefficient, then add the four results pixel by pixel. Compare your answer with the original image.

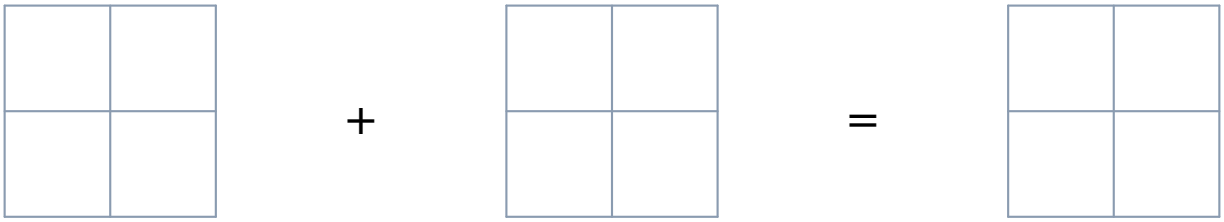
c_1 · b1		**c_2 · b2**		**c_3 · b3**		**c_4 · b4**		**Result**

Does your reconstruction match the original exactly? _____

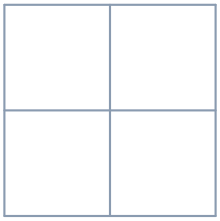
Lossy reconstruction

Now keep only the two largest coefficients, c_1 and c_2, and discard the other two. Rebuild the image from just those two terms.

$c_1 \cdot b1$ + **$c_2 \cdot b2$** = **Approximation**



Error (original minus approximation), pixel by pixel:



Squared error (sum of squared errors) = _____

$4 \times (c_3^2 + c_4^2) =$ _____

What do you notice? _____

Part 2: A 3×3 Image with the Frei-Chen Basis

A 3×3 image has nine pixels, so a basis needs nine pattern images. The Frei-Chen basis was designed with meaning in mind: four patterns respond to edges, four respond to lines, and one is the average.

The basis

Each mask is shown with integer-ish entries, followed by the number you divide by to give it length 1.

G1 (÷ 2√2)

1	√2	1
0	0	0
-1	−√2	-1

G2 (÷ 2√2)

1	0	-1
√2	0	−√2
1	0	-1

G3 (÷ 2√2)

0	-1	√2
1	0	-1
−√2	1	0

G4 (÷ 2√2)

√2	-1	0
-1	0	1
0	1	−√2

Edge patterns (G1–G4)

G5 (÷ 2)

0	1	0
-1	0	-1
0	1	0

G6 (÷ 2)

-1	0	1
0	0	0
1	0	-1

G7 (÷ 6)

1	-2	1
-2	4	-2
1	-2	1

G8 (÷ 6)

-2	1	-2
1	4	1
-2	1	-2

G9 (÷ 3)

1	1	1
1	1	1
1	1	1

Line patterns (G5–G8) and average (G9)

This basis is *orthonormal*: each coefficient is just the dot product with the normalized mask, with no extra division at the end.

The image

A bright horizontal line on a dark background:

Image

0	0	0
9	9	9
0	0	0

Compute the coefficients

Tip: take the dot product with the integer mask, then divide by the mask's divisor. Look at which rows of the image are nonzero before you start; it can save you a lot of work.

G1 coefficient = _____

G2 coefficient = _____

G3 coefficient = _____

G4 coefficient = _____

G5 coefficient = _____

G6 coefficient = _____

G7 coefficient = _____

G8 coefficient = _____

G9 coefficient = _____

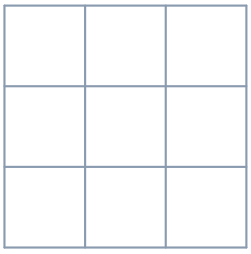
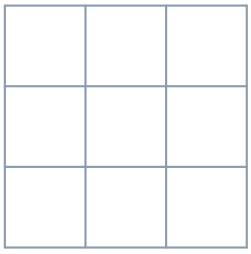
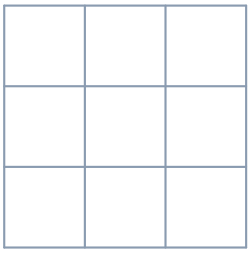
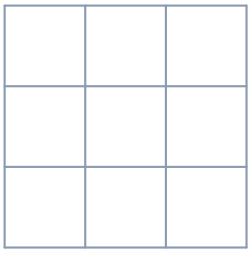
Which coefficients are nonzero? What does that tell you about the image?

_____

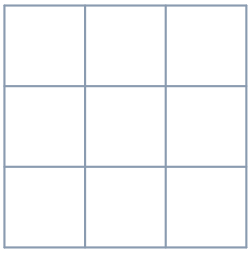
_____

Lossless reconstruction

Keep only the nonzero coefficients. Multiply each one by its *normalized* basis image (the mask divided by its divisor), then add the results. Label each grid with the coefficient and basis image you used. You may not need every grid.

___ × G__  + **___ × G__**  + **___ × G__**  + **___ × G__** 

Reconstructed

= 

Does your reconstruction match the original exactly? _____

Work space